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LETTER TO THE EDITOR

Monodromy of a two degrees of freedom Liouville integrable system with many focus–focus singular points

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Abstract

This letter deals with the global monodromy of singular Lagrangian toral fibrations defined by two degrees of freedom Liouville integrable systems with only focus–focus singular points. We show that any global monodromy matrix in $Sl(2, \mathbb{Z})$ is realizable by such a system.

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Recently, there has been an increase of interest in classical integrable systems with monodromy [1, 2] because of its presence in quantum systems [3–5]. At present only examples with the simplest monodromy have been described [6–9]. In this letter, we show that a two degrees of freedom Liouville integrable Hamiltonian system having a sufficiently large number of focus–focus critical points, each of which possesses an elementary local monodromy matrix, can have an arbitrary global monodromy matrix. In particular, the minimal number of focus–focus points needed to realize the Arnold's cat map is shown to be equal to 12. This number is also needed to realize the identity as a global monodromy matrix. We note that the global monodromy matrix is a rather crude invariant of two degrees of freedom Liouville integrable systems as it does not distinguish between no focus–focus critical points and the presence of groups of 12 such critical points.

We now give a formal mathematical statement of our result, which we cannot find explicitly mentioned in the literature. Let $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Consider the set $\mathcal{S}$ formed by finite products of conjugates of T by elements of $Sl(2, \mathbb{Z})$, the group of 2×2 integer matrices with determinant 1. In other words,

$$\mathcal{S} = \left\{ \prod_{i=1}^{n} A_i T A_i^{-1} \in Sl(2, \mathbb{Z}) \,\middle|\, A_i \in Sl(2, \mathbb{Z}) \text{ and } n \in \mathbb{Z}_{\geqslant 1} \right\}. \qquad (1)$$

In (1), the matrix A_i is allowed to be $\pm I = \pm\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$; so $T^k \in \mathcal{S}$ for every $k \in \mathbb{Z}_{\geqslant 1}$.

Theorem. $\mathcal{S} = Sl(2, \mathbb{Z})$.

We will return to proving this theorem after we explain in more detail the origin of the set $\mathcal{S}$ in the study of two degrees of freedom Liouville integrable systems with only focus–focus singular points. We recall the global geometric formulation of such systems [10].

Let M be a four-dimensional smooth connected symplectic manifold with symplectic form ω. Let B be an oriented two-dimensional smooth manifold. Suppose that $F : M \to B$ is a smooth proper surjective map with connected Lagrangian fibres, which is a submersion except for a discrete number of critical values $c_i \in B$. Assume that each critical point of F in the fibre $F^{-1}(c_i)$ is a focus–focus singularity. This means that at $m \in M$, where $\mathrm{d}F(m) = 0$, there is a chart (U, φ) of M at m with local canonical coordinates (x, y, ξ, η), where $\omega|U = \mathrm{d}x \wedge \mathrm{d}\xi + \mathrm{d}y \wedge \mathrm{d}\eta$, and a chart (V, ψ) of B at $F(m)$ such that the vector space spanned by the Hessians $D^2F_1(m)$ and $D^2F_2(m)$, where (F_1, F_2) are the components of F in the given charts, is generated by the quadratic forms

$$q_1 = x\xi + y\eta \qquad \text{and} \qquad q_2 = x\eta - y\xi.$$

This is the Williamson normal form for the focus–focus singularity [11].

Under the above hypotheses, the (semiglobal) monodromy theorem [12] holds. Namely, there is a suitable oriented closed 2-disc $\overline{D}_i \subseteq \mathbb{R}^2$, centred at c_i, such that $c_j \notin \overline{D}_i$ if $j \neq i$. Since $\overline{D}_i^* = \overline{D}_i \backslash \{c_i\}$ lies in the set of regular values of F, the map $\pi_i : F^{-1}(\partial\overline{D}_i) \to \partial\overline{D}_i$ is a smooth fibration with fibre which is a smooth two-dimensional torus. Let k_i be the number of focus–focus critical points in the critical fibre $F^{-1}(c_i)$. Then choosing a suitable ordered basis of the first integer homology group $H_1\big(F^{-1}(b_i), \mathbb{Z}\big)$ of the fibre $F^{-1}(b_i)$, where $b_i \in \partial\overline{D}_i$, the (local) monodromy of the fibration π_i is $\mathcal{M}_i = \left(\begin{smallmatrix} 1 & k_i \\ 0 & 1 \end{smallmatrix}\right)$. We note that there are other ways of forming a singular fibre than just pinching several cycles, which are equal homologically, to a point. (See the statement of the monodromy theorem.) But we will not discuss them here.

From the geometric version of the action-angle coordinate theorem [2], the oriented smooth manifold $B_r = B \backslash \{c_i\}$ has an integral affine structure. The affine structure means that there is an affine connection $\widetilde{\nabla}$ on T^*B_r, which is flat and torsion free [13]. The integral structure may be described as follows. There are two 1-forms α_i on B_r, which are linearly independent and are $\widetilde{\nabla}$-parallel along the integral curves of the vector fields V_j defined by $V_j \,\lrcorner\, \alpha_i = \delta_{ij}$.[3] Moreover, the Hamiltonian vector fields $\omega^\sharp(F^*\alpha_i)$ (see footnote 4) on (M, ω) when restricted to the torus $F^{-1}(b_0)$ have period 1 and form a $\mathbb{Z}$-basis of $H_1\big(F^{-1}(b_0), \mathbb{Z}\big)$ for every $b_0 \in B_r$.

Suppose that we fix an index i_0. Let $\gamma_{i_0} : S^1 \to B_r$ be an oriented loop whose image is the circle $\partial\overline{D}_{i_0}$ which surrounds only the critical value c_{i_0}. Let $b_{i_0} = \gamma_{i_0}(e)$ and choose an ordered basis $\big\{X_1\big(b_{i_0}\big), X_2\big(b_{i_0}\big)\big\}$ of $T_{b_{i_0}}B_r$ such that after parallel translation along γ_{i_0} by the flat affine connection ∇ on B_r, which is dual to the connection $\widetilde{\nabla}$, we obtain the holonomy matrix $\mathcal{M}_{i_0}$. Suppose that γ_i is another such loop around c_i with $i \neq i_0$. Let $\Gamma_{i_0 i}$ be an oriented path in B_r which joins b_i to b_{i_0}. Then using the connection ∇ to parallel transport the frame $\big\{X_1\big(b_{i_0}\big), X_2\big(b_{i_0}\big)\big\}$ along the closed curve $\Gamma_i = \Gamma_{i_0 i}\circ\gamma_i\circ\big(\Gamma_{i_0 i}\big)^{-1}$ gives the holonomy $\mathcal{N}_i \in Gl(2, \mathbb{R})$. Since $\mathcal{N}_i$ is an isomorphism of $H_1\big(F^{-1}(b_i), \mathbb{Z}\big)$, it is an element of $Gl(2, \mathbb{Z})$. However, because parallel transport is orientation preserving, $\mathcal{N}_i$ lies in $Sl(2, \mathbb{Z})$.

[3] We have used the notation $\lrcorner$ of Sternberg [14] to denote the contraction of a vector field with a 1-form, namely, $(V_j \,\lrcorner\, \alpha_i)(b) = \alpha_i(b)(V_j(b))$.

[4] Here we follow the convention of Abraham–Marsden [15] to define the map $\omega^\sharp(b) : T_b^*B_r \to T_bB_r$ by $\omega(b)(\omega^\sharp(b)(\alpha_b), v_b) = \alpha_b(v_b)$ for every $b \in B_r$.

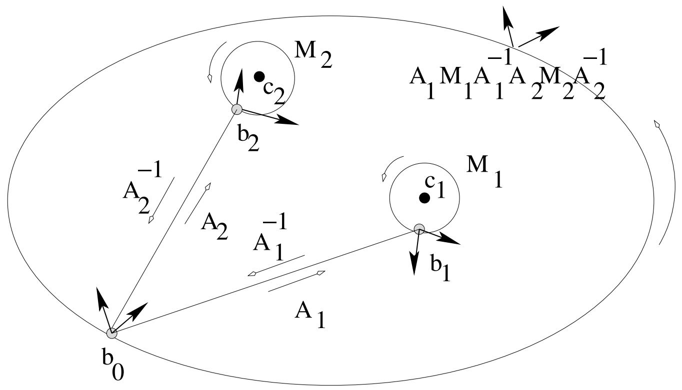

Figure 1. Relation between local monodromy matrices and global monodromy.

There is a flat affine connection $\widehat{\nabla}$ on the cohomology bundle $H^*(F_r) : H^1\big(F^{-1}(B_r), \mathbb{Z}\big) \to B_r$ associated with the fibration $F_r = F|F^{-1}(B_r) : F^{-1}(B_r) \to B_r$, which is induced by the connection $\widetilde{\nabla}$. We note that the cohomology bundle is nothing but the bundle of period lattices. Because F_r is a proper map, the connection $\widehat{\nabla}$ is complete. Hence we can parallel transport the fibres of F_r along the curve $\Gamma_{i_0 i}$. This gives a linear bijection A_i of $H^1\big(F_r^{-1}(b_i), \mathbb{Z}\big)$ into $H^1\big(F_r^{-1}\big(b_{i_0}\big), \mathbb{Z}\big)$. Using $\widehat{\nabla}$ to parallel transport along the closed curve Γ_i we obtain the holonomy matrix $N_i = A_i \mathcal{M}_i A_i^{-1}$. Because the connection $\widehat{\nabla}$ is induced by the connection $\widetilde{\nabla}$, it follows that $N_i = \mathcal{N}_i$. Hence $\mathcal{N}_i = A_i \mathcal{M}_i A_i^{-1}$, where $A_i \in Sl(2, \mathbb{Z})$.

If we now take a loop Γ in B_r with base point b_{i_0}, which encloses the set $\big\{c_{i_1}, \ldots, c_{i_n}\big\}$ of critical values of the map F, then Γ is homotopic in B_r to the curve $\Gamma_{i_n} \circ \cdots \circ \Gamma_{i_1}$. The holonomy of the frame $\big\{X_1\big(b_{i_0}\big), X_2\big(b_{i_0}\big)\big\}$ around Γ is $\prod_{j=1}^{n} A_{i_j} \mathcal{M}_{i_j} A_{i_j}^{-1}$. This explains the origin of the set $\mathcal{S}$ introduced at the beginning of this letter. Figure 1 illustrates this last argument in a schematic way.

A recent result of Zung [16] states that any affine structure on B (in particular, any discrete collection of focus–focus critical values with possibly multiple focus–focus singular points in the same fibre) can be realized by a suitable map $F : M \to B$ which satisfies our assumptions.

We now return to the proof of the theorem formulated at the beginning of this letter and then to a discussion of its consequences.

Proof of the theorem. As is well known the group $Sl(2, \mathbb{Z})$ has the presentation

$$\langle U, V; U^4 = I, U^2 = (UV)^3 \rangle$$

see Serre [17]. Thus to prove the theorem it suffices to give an explicit construction of the generators U and V as products of $Sl(2, \mathbb{Z})$-conjugates of $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Let $B = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Then

$$U = T(BTB^{-1})T \tag{2}$$

$$V = (BTB^{-1})\,T[T(BTB^{-1})T]^3 \tag{3}$$

satisfy $U^4 = 1$ and $U^2 = (UV)^3$. □

Remark. Compare this argument with the construction of elements of $Sl(2, \mathbb{Z})$ given in [18–20].

The above theorem has several consequences that are important for applications.

(1) The element V is nothing but T^{-1}. V is a product of *eleven* $Sl(2,\mathbb{Z})$-conjugates of T.

(2) The element UV can be written as product of *two* $Sl(2,\mathbb{Z})$-conjugates of T. Namely, let $C = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$. Then $UV = (CTC^{-1})(BTB^{-1})$.

(3) The identities $U^4 = 1$ and $(UV)^6 = 1$ are a product of *twelve* $Sl(2,\mathbb{Z})$-conjugates of T.

(4) Arnold's cat map $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ can be expressed as a product of *twelve* $Sl(2,\mathbb{Z})$-conjugates of T. Explicitly,

$$\left[\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} V \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right]\left[\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} T \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\right] = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}. \tag{4}$$

(5) There are hyperbolic maps which have a much shorter presentation as products of $Sl(2,\mathbb{Z})$-conjugates of T than Arnold's cat map. For example, hyperbolic matrices with trace $2-k^2$ with k an integer greater than two can be written as

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\left[\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -k & 1 \end{pmatrix}\right] = \begin{pmatrix} 1-k-k^2 & 2+k \\ -k^2 & 1+k \end{pmatrix}. \tag{5}$$

We now give a formulation of the above group theoretical results in terms of two degrees of freedom Liouville integrable systems with multiple focus–focus points.

Corollary. *Every element of $Sl(2,\mathbb{Z})$ may be obtained as the monodromy of a two degrees of freedom Liouville integrable system whose momentum map is proper, has connected fibres, and has a discrete number of critical values where the corresponding singular fibre has only focus–focus singular points possibly with multiplicity.*

In order to construct an explicit example of a dynamical system with any given monodromy matrix it may be useful to know the minimal number of singular fibres (probably with nontrivial multiplicity) needed to realize such a system. The authors do not know the answer to this general question. We have found that to realize Arnold's cat map as the monodromy of a two degrees of freedom Liouville integrable system whose momentum mapping is proper, we need three critical focus–focus fibres of multiplicity 9, 2 and 1 respectively. Namely, let us introduce matrices $X = \begin{pmatrix} -2 & -1 \\ 1 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} -4 & -1 \\ 1 & 0 \end{pmatrix}$, then we have

$$[T^9][YTY^{-1}][XT^2X^{-1}] = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}. \tag{6}$$

The above argument shows the possibility of improving Boslinov's and Dullin's example [21] of a three degrees of freedom Liouville integrable system with Arnold's cat map as monodromy by reducing the number of degrees of freedom to two. Moreover, in order to study quantum and classical systems with hyperbolic monodromy maps it would probably be better to start with hyperbolic maps which have a simple presentation in terms of multiple focus–focus points. As we have shown in (5) above, an integrable system with two focus–focus points can lead to a hyperbolic monodromy matrix.

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Added in proofs. Dr. V Mateev kindly pointed us to the reference [22] which contains mathematical statements very close to the theorem formulated in our letter.

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